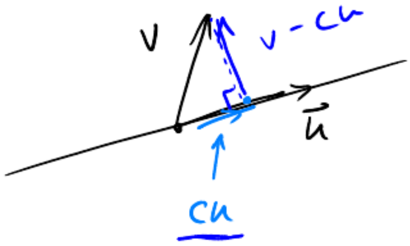
ORTHOGONALITY

Recall: Vectors u, v in $\mathbb{R}^n$ are orthogonal (or perpendicular) when $u \cdot v = 0$.

(idea: $u \cdot v = 0 \Rightarrow 0 = u \cdot v = |u||v|\cos(\theta)$
so provided $u \neq \vec{0} \neq v$, we see $\cos(\theta) = 0 \Rightarrow \theta = \frac{\pi}{2}$)

Q: Can we project vectors orthogonally?
i.e. Can we measure "how far v tends in direction of u"?

A: Yes!



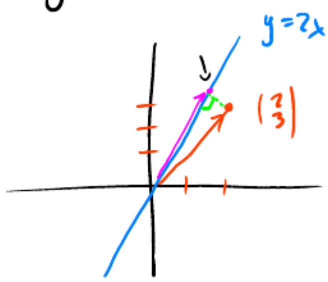
Derivation: Given two vectors $u, v \in \mathbb{R}^n$ w/ $\boxed{u \neq \vec{0}}$. We seek a vector cu w/ $v - cu$ is orthogonal to u.

i.e. $u \cdot (v - cu) = 0$ so $u \cdot v - c(u \cdot u) = 0$, which yields $c(u \cdot u) = u \cdot v$, so $c = \frac{u \cdot v}{u \cdot u}$ noting $u \cdot u \neq 0$.

Hence $$\boxed{\mathrm{proj}_{\mathrm{span}(u)}(v) = cu = \left(\frac{u \cdot v}{u \cdot u}\right) u}$$ ☺.

↑ projection of v onto the span of u.

Ex: Compute the projection of $\binom{2}{3}$ onto the line $y = 2x$ in $\mathbb{R}^2$



Sol: We choose a vector in the direction of the line $y = 2x$:

$$\ell = \left\{\binom{x}{y} : y = 2x\right\} = \left\{\binom{x}{2x} : x \in \mathbb{R}\right\} = \mathrm{span}\left\{\binom{1}{2}\right\}.$$

$$\therefore \mathrm{proj}_\ell\binom{2}{3} = \frac{\binom{2}{3}\cdot\binom{1}{2}}{\binom{1}{2}\cdot\binom{1}{2}}\binom{1}{2} = \frac{2+6}{1+2^2}\binom{1}{2} = \frac{8}{5}\binom{1}{2}. \quad \boxtimes$$

Ex: Compute the orthogonal projection of $\begin{pmatrix}1\\2\\1\\3\end{pmatrix}$ onto $\mathrm{span}\left\{\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}\right\}$.

Sol: $v = \begin{pmatrix}1\\2\\1\\3\end{pmatrix}$, $u = \begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$ so $\mathrm{proj}_{\mathrm{span}(u)}(v) = \frac{u \cdot v}{u \cdot u} u$

$u \cdot v = -1 - 2 - 1 + 3 = 3$
$u \cdot u = (-1)^2 + 1^2 + (-1)^2 + 1^2 = 4$ and thus $\text{proj}_{\text{span}(u)}(v) = \frac{3}{4} u = \frac{3}{4}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$. $\boxed{//}$

<u>Defn</u>: A collection $\{v_1, v_2, \cdots, v_n\}$ is <u>pairwise orthogonal</u> (aka <u>mutually orthogonal</u>) when every pair of distinct vectors v_i, v_j is an orthogonal pair. I.e. for all $1 \le i < j \le n$ we have $v_i \cdot v_j = 0$.

<u>Ex</u>: $\mathcal{E}_n \leftarrow$ the standard basis on $\mathbb{R}^n$ is a pairwise orthogonal collection.

$$e_i \cdot e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

<u>Ex</u>: $\left\{ \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ are <u>not</u> mutually orthogonal.

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 4 + 6 = 10 \ne 0 \ldots$$

<u>Q</u>: Can we modify the collection to build a mutually orthogonal one?

<u>Prop</u>: If $S = \{v_1, v_2, \cdots, v_n\}$ is a collection of pairwise orthogonal <u>nonzero</u> vectors, then S is lin. indep.

<u>pf</u>: Assume S is a collection of pairwise orthogonal nonzero vectors, and suppose $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}$. Now $v_i \cdot v_j = 0$ when $i \ne j$, and nonzero when $i = j$. Hence:

$$v_i \cdot (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = v_i \cdot \vec{0} = 0$$

OTOH: $v_i \cdot (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 (v_i \cdot v_1) + c_2 (v_i \cdot v_2) + \cdots + c_n (v_i \cdot v_n)$

$$= c_1 0 + c_2 0 + \cdots + c_i (v_i \cdot v_i) + \cdots + c_j 0$$

$$= 0 + 0 + \cdots + c_i (v_i \cdot v_i) + \cdots + 0$$

$$= c_i (v_i \cdot v_i).$$

So $0 = c_i (v_i \cdot v_i)$, and $v_i \cdot v_i \ne 0$ because $v_i \ne \vec{0}$; thus $c_i = 0$.

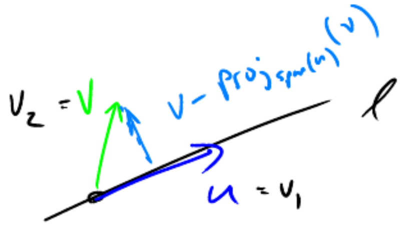
Hence $c_i = 0$ for all $1 \le i \le n$, and we see S is lin. ind. $\boxed{//}$

<u>Point</u>: Mutually orthogonal nonzero vectors are linearly independent :).

Cor: If S is a collection of n mutually orthogonal vectors in $\mathbb{R}^n$, then S is a basis for $\mathbb{R}^n$.

Returning to the example from before: $S = \left\{ v_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$.

Goal: Build a collection $\hat{S}$ of vectors based on S which is a mutually orthogonal collection...



Start Building $\hat{S}$: $\hat{S}_1 = \{ u_1 = v_1 \}$

Let $u_2 = v_2 - \text{proj}_{\text{span}(u_1)}(v_2)$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\text{proj}_{\text{span}(u_1)} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 = \frac{10}{4^2 + 2^2} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{10}{20} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Let $\hat{S}_2 = \{ u_1, u_2 \}$. **Claim:** $\hat{S}$ is mutually orth. coll.

Check: $u_1 \cdot u_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 4 \cdot (-1) + 2 \cdot 2 = 0$

Point: Projections allow us to build mutually orthogonal collections of vectors from arbitrary lin. indep collections in $\mathbb{R}^n$.

Q: How important was the fact we had only two vectors?

Ex: Consider the basis $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$ for $\mathbb{R}^3$.
$\quad v_1 \qquad v_2 \qquad v_3$

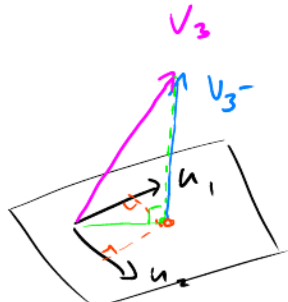
$\ast$ $u_1 = v_1$

$\ast$ $u_2 = v_2 - \text{proj}_{\text{span}(u_1)}(v_2) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \dfrac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \dfrac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{2}{3} \end{pmatrix}$

$$= \frac{2}{3} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

NB: For a basis of orthogonal vectors, the representation of every $w \in \mathbb{R}^n$ w.r.t. the orthogonal basis is determined by the dot product w/ each vector of the basis...

$P = \text{span}(u_1, u_2)$



$$u_3 = v_3 - \underline{\mathrm{proj}_{\mathrm{span}(u_2)}(v_3)} - \underline{\mathrm{proj}_{\mathrm{span}(u_1)}(v_3)}$$

$v_3 = \begin{pmatrix}1\\0\\3\end{pmatrix}$

$u_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \; u_2 = \frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}$

$$= v_3 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2 - \frac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1$$

$$= \begin{pmatrix}1\\0\\3\end{pmatrix} - \frac{\frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\3\end{pmatrix}}{\frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}\cdot\frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}} \frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix} - \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\3\end{pmatrix}}{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\3\end{pmatrix} - \left(\frac{-1-3}{1+4+1}\right)\begin{pmatrix}-1\\2\\-1\end{pmatrix} - \frac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\3\end{pmatrix} + \frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix} - \frac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}3-2-4\\0+4-4\\9-2-4\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-3\\0\\3\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Check: $\hat{S} = \{u_1, u_2, u_3\}$ is a collection of mutually orthogonal vectors.

$$u_1 \cdot u_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix} = \frac{2}{3}(-1+2-1) = 0$$

$$u_1 \cdot u_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = -1+0+1 = 0$$

$$u_2 \cdot u_3 = \frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = \frac{2}{3}(1+0-1) = 0$$

$\boxtimes$

Prop (Gram-Schmidt Process): Let $v_1, v_2, \dots, v_k$ be a sequence of linearly independent vectors in $\mathbb{R}^n$. The sequence

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1$$

$$\vdots$$

$$u_k = v_k - \frac{u_{k-1} \cdot v_k}{u_{k-1} \cdot u_{k-1}} u_{k-1} - \frac{u_{k-2} \cdot v_k}{u_{k-2} \cdot u_{k-2}} u_{k-2} - \cdots - \frac{u_1 \cdot v_k}{u_1 \cdot u_1} u_1$$

is a collection of mutually orthogonal vectors. $\boxtimes$